Everything, Entangled, All At Once

Making Sense of Non-Locality and the Jaynes-Cummings Model

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Outline

The Language of Quantum: The Qubit

A qubit is the fundamental unit of quantum information.

- A classical bit is a macroscopic system that can be in one of two states: 0 or 1.
- A qubit is a microscopic quantum system that can exist in a superposition of both states simultaneously.

Dirac Notation (Bra-Ket):

- A column vector is a "ket": $|\psi\rangle$. E.g., $|0\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|1\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
- A row vector (Hermitian conjugate) is a "bra": $\langle\psi| = |\psi\rangle^\dagger$. E.g., $\langle 0| \equiv \begin{pmatrix} 1 & 0 \end{pmatrix}$.
- The inner product is a "bra-ket": $\langle\phi|\psi\rangle$.

A general qubit state is written as:

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$$

Where $|\alpha|^2 + |\beta|^2 = 1$.

Describing Two Qubits: The Tensor Product

To describe a composite system of multiple qubits, we use the tensor product ($\otimes$).
If Qubit A is $|\psi_A\rangle$ and Qubit B is $|\psi_B\rangle$, the combined system is $|\Psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$.

Vector Representation: The tensor product combines vector spaces: $\mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^4$.

$$|01\rangle \equiv |0\rangle_A \otimes |1\rangle_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

The four basis states for a two-qubit system are:

- $|00\rangle$: Both qubits are in state 0.
- $|01\rangle$: Qubit A is 0, Qubit B is 1.
- $|10\rangle$: Qubit A is 1, Qubit B is 0.
- $|11\rangle$: Both qubits are in state 1.

Separable vs. Entangled States

A two-qubit state is **separable** (or a product state) if it can be written as a tensor product of individual qubit states: $|\Psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$.

A state is entangled if it is **not separable**.

Example: Proving a Bell State is Entangled

Consider the Bell state $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Can this be factored? Assume it is separable: $|\Phi^+\rangle = (\alpha_A |0\rangle + \beta_A |1\rangle) \otimes (\alpha_B |0\rangle + \beta_B |1\rangle)$. Expanding this gives:

$$\alpha_A\alpha_B |00\rangle + \alpha_A\beta_B |01\rangle + \beta_A\alpha_B |10\rangle + \beta_A\beta_B |11\rangle$$

Comparing coefficients with the Bell state, we see that $\alpha_A\beta_B = 0$ and $\beta_A\alpha_B = 0$. This leads to a contradiction, as it would require the coefficients of $|00\rangle$ or $|11\rangle$ to be zero.

The Archetype of Entanglement: The Bell States

The four **Bell states** are maximally entangled two-qubit states that form a complete basis.

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \qquad |\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \qquad |\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

Physical Meaning of $|\Phi^+\rangle$**:**

- The system is in a superposition of "both 0" and "both 1".
- If Alice measures her qubit and gets **0**, she instantly knows Bob's qubit is **0**.
- If she measures **1**, she instantly knows Bob's is **1**.

The outcomes are individually random, but perfectly correlated.

Measurement: The Projection Postulate

Measurement is an irreversible process that links a quantum system to a classical observer, projecting a quantum state onto a definite outcome.

Projective Measurement: A measurement in an orthonormal basis $\{|\phi_i\rangle\}$ is described by a set of projection operators $\hat{P}_i = |\phi_i\rangle \langle\phi_i|$.

- The probability of getting outcome 'i' for a state $|\Psi\rangle$ is:

$$p(i) = \langle\Psi| \hat{P}_i |\Psi\rangle = |\langle\phi_i|\Psi\rangle|^2$$

- If outcome 'i' is obtained, the state of the system "collapses" to:

$$|\Psi'\rangle = \frac{\hat{P}_i |\Psi\rangle}{\sqrt{p(i)}} = |\phi_i\rangle$$

This is the "collapse of the wavefunction." The measurement forces the system into one of the basis states.

Projective Measurement: An Example

Let's measure the state $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$ in the standard $\{|0\rangle, |1\rangle\}$ basis.
The projection operators are:

$$\hat{P}_0 = |0\rangle \langle 0| \quad \text{and} \quad \hat{P}_1 = |1\rangle \langle 1|$$

- **Probability of measuring '0':**

$$p(0) = \langle\psi| \hat{P}_0 |\psi\rangle = \langle\psi| |0\rangle \langle 0|\psi\rangle = | \langle 0|\psi\rangle |^2 = |\alpha|^2$$

- **State after measuring '0':**

$$|\psi'\rangle = \frac{\hat{P}_0 |\psi\rangle}{\sqrt{p(0)}} = \frac{|0\rangle \langle 0|\psi\rangle}{|\alpha|} = \frac{\alpha |0\rangle}{|\alpha|} \equiv |0\rangle$$

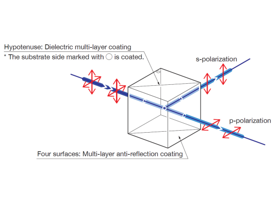
The measurement projects the state vector onto the basis axes, and the squared lengths give the probabilities.

How Are Measurements Performed in the Lab?

A theoretical projection is realized by a physical apparatus that couples the qubit's state to a macroscopic outcome.

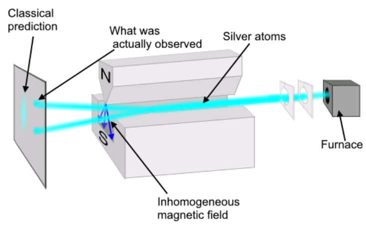
Photon Polarization:

- A **polarizing beam splitter (PBS)** transmits horizontal ($|H\rangle$) and reflects vertical ($|V\rangle$) photons.
- Single-photon detectors at the outputs click for a specific outcome.



Electron/Atom Spin:

- A **Stern-Gerlach apparatus** uses an inhomogeneous magnetic field.
- The field deflects the particle "up" ($|\uparrow\rangle$) or "down" ($|\downarrow\rangle$) based on its spin.



Collapse in Action: Measuring a Bell State

Let's apply the measurement postulate to our entangled state $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.
The projection operator for Alice measuring '0' on her qubit is $\hat{P}_{0,A} \otimes I_B$.
Probability of Alice measuring '0': $p(0) = \langle\Phi^+| (\hat{P}_{0,A} \otimes I_B) |\Phi^+\rangle = \frac{1}{2}$

State Collapse after Alice measures '0': The new state is:

$$|\Psi'\rangle = \frac{(\hat{P}_{0,A} \otimes I_B) |\Phi^+\rangle}{\sqrt{p(0)}} = \frac{\frac{1}{\sqrt{2}} |00\rangle}{\sqrt{1/2}} = |00\rangle$$

Bob's qubit is now definitely in the state $|0\rangle$.

This collapse happens instantly for the entire system, no matter the distance. This is the source of the "spookiness".

Intuition from the Classical World: Locality

Our everyday intuition is built on the principle of **locality**.

Principle of Locality

An object is only directly influenced by its immediate surroundings. An influence cannot travel faster than the speed of light.

Example: A Ripple in a Pond

- If you drop a stone at point A, a person at point B won't feel the ripple instantly.
- The wave has to travel across the water's surface. The effect is local.

The instantaneous collapse of the wavefunction seems to violate this fundamental principle.

A Classical Explanation? Realism & Hidden Variables

How to explain entanglement's perfect correlations without "spooky action"? Assume the information was there all along.

The Glove Analogy (Local Hidden Variables)

A magician puts a pair of gloves into two boxes. One box contains the left glove, one the right. He gives one to Alice and one to Bob.

- When Alice opens her box and finds a **left-handed glove**, she knows instantly that Bob has the **right-handed glove**.
- This is not spooky. The "outcome" was determined from the start. This predetermined information is a **"hidden variable."**

This worldview is called Local Realism.

- **Realism:** Objects have definite properties independent of measurement.
- **Locality:** Actions at one point cannot instantly affect another distant point.

The EPR Paradox (1935)

Einstein, Podolsky, and Rosen (EPR) used Local Realism to argue that quantum mechanics was incomplete.

The EPR Argument

If Alice can measure her particle and know Bob's property without touching it, that property must have been real and pre-determined all along. Quantum mechanics, which says properties are indefinite until measured, must be incomplete. There must be local hidden variables (LHV) that pre-determine all outcomes, just like the gloves.[1]

For 30 years, this was a philosophical debate. Is the quantum "spookiness" real, or an illusion covering up for classical, predetermined information?

Bell's Theorem (1964): The Deciding Test

John Bell devised a theorem that could experimentally distinguish between quantum mechanics and any local hidden variable theory.

- He showed that correlations from any LHV theory must obey a **Bell Inequality**.
- He then showed that quantum mechanics predicts this inequality will be violated.
- This moved the debate from philosophy to experimental physics.

Is the universe locally real,
or is it quantum mechanical?

Bell's theorem provided a way to ask nature the question directly.[2]

The CHSH Inequality: A Concrete Test

A common testable version is the **CHSH inequality**. Alice chooses to measure A or A', Bob chooses B or B'. The results are ± 1.

Prediction from Local Realism

Any local hidden variable theory must satisfy:

$$S = |E(A,B) - E(A,B')| + |E(A',B) + E(A',B')| \leq 2$$

where $E(A,B)$ is the correlation. [3]

Prediction from Quantum Mechanics

Quantum mechanics predicts this inequality will be violated. For an entangled state, quantum theory predicts S can reach a maximum value of:

$$S_{QM} = 2\sqrt{2} \approx 2.828$$

The CHSH Game: A Test of Reality

The CHSH inequality can be framed as a cooperative game played by Alice and Bob against a Referee. This makes the physics more intuitive.[9]

Nonlocal Games

- Alice and Bob are cooperative players. They can plan a strategy beforehand.
- Once the game starts, they are **forbidden from communicating**.
- A Referee gives Alice a random question bit x and Bob a random question bit y.
- Alice must return an answer bit a, and Bob an answer bit b.
- They win or lose based on a public set of rules connecting (x, y) to (a, b).

A Non-Local Game

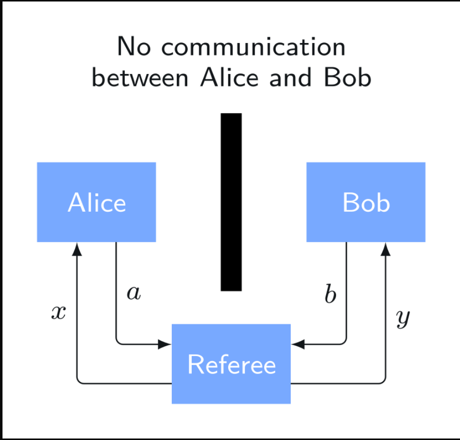


Figure: A Schematic of a Non-local Game

CHSH Game: The Rules

The specific rules for the CHSH game are as follows:

- The questions $x, y \in \{0, 1\}$ are chosen by the referee uniformly at random.
- The answers $a, b \in \{0, 1\}$ must satisfy the winning condition:

$$a \oplus b = x \wedge y$$

($\oplus$ is addition mod 2, $\wedge$ is the logical AND).

Winning Conditions Table

Questions (x,y)	Win if...	Lose if...
(0,0)	$a = b$	$a \neq b$
(0,1)	$a = b$	$a \neq b$
(1,0)	$a = b$	$a \neq b$
(1,1)	$a \neq b$	$a = b$

The challenge: Alice doesn't know y, and Bob doesn't know x.

Limitation of Classical Strategies

How well can Alice and Bob do using only classical info (even with shared random bits)?

Best Deterministic Strategy

Consider a strategy where their answers are functions of their questions: $a(x)$ and $b(y)$.

- To win for (0,0), they need $a(0) = b(0)$.
- To win for (0,1), they need $a(0) = b(1)$.
- To win for (1,0), they need $a(1) = b(0)$.

These three conditions together imply $a(1) = b(0) = a(0) = b(1)$. So, $a(1) = b(1)$.

But to win for (1,1), they need $a(1) \neq b(1)$. This is a **contradiction**. No classical strategy can win all four cases. The best they can do is win 3 out of 4 times.

Maximum Classical Success Probability

The best classical strategy (deterministic or probabilistic) wins with a probability of **75%**.

The Quantum Advantage

What if Alice and Bob prepare a shared entangled state before the game starts?

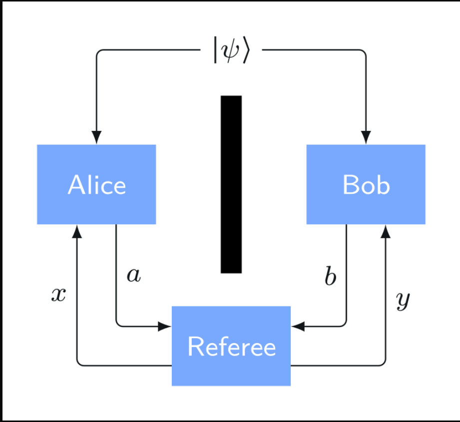


Figure: A Non-Local Game with Entanglement [9]

Quantum Strategy

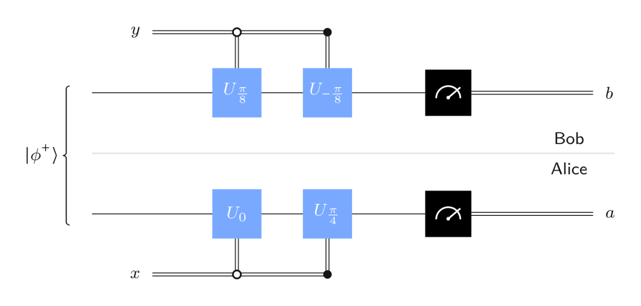
Quantum Strategy

1. **Setup:** Alice and Bob share the Bell state $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.
2. **Actions:** Depending on their question bit (x for Alice, y for Bob), they each perform a specific rotation on their own qubit.
3. **Answers:** They measure their qubit in the $\{|0\rangle, |1\rangle\}$ basis and return the outcome as their answer.

The Winning Quantum Strategy

The specific rotations are chosen carefully.

- **Alice's action on her qubit:** If $x = 0$, she rotates by angle 0. If $x = 1$, she rotates by angle $\pi/4$.
- **Bob's action on his qubit:** If $y = 0$, he rotates by angle $\pi/8$. If $y = 1$, he rotates by angle $-\pi/8$.



The Winning Quantum Strategy II

The Result of this Strategy

A full analysis shows that for every possible question pair (x, y), this strategy gives Alice and Bob a winning probability of:

$$P(\text{win}) = \cos^2(\pi/8) = \frac{2+\sqrt{2}}{4} \approx 85.4\%$$

This is significantly better than the 75% classical limit and is the maximum allowed by quantum mechanics (Tsirelson's bound).

CHSH Game: The Final Word

- The CHSH game is a concrete, testable demonstration of Bell's theorem.
- It shows that quantum mechanics is incompatible with local hidden variable theories. The correlations predicted and observed are stronger than any classical theory can explain.
- Experiments implementing the CHSH game have been performed many times, and the results consistently violate the classical bound and agree with quantum predictions.

A Nobel Prize-Winning Idea

Observing entanglement through Bell tests, like the CHSH game, provides such profound insight into the nature of reality that the 2022 Nobel Prize in Physics was awarded to Alain Aspect, John Clauser, and Anton Zeilinger for their pioneering experiments in this area.

Experiment 1: Aspect's Groundbreaking Test (1982)

Alain Aspect's team performed one of the first and most influential tests of Bell's inequality. [4]

- **Source:** A cascade in Calcium-40 atoms produced polarization-entangled photons.
- **Setup:** Photons were sent to two polarizers 12 meters apart.
- **Closing the Locality Loophole:** A fast switch changed polarizer settings in 10 ns, faster than the 40 ns light-travel time between them. This ensured the setting at one end could not influence the other.
- **Result:** They measured $S = 2.697 \pm 0.015$. This clearly violates the LHV limit of 2 and agrees with quantum mechanics.

This was a major blow to local realism.

Lingering Doubts: Experimental Loopholes

Aspect's experiment was fantastic, but determined skeptics could still point to potential loopholes.

1. Locality (or Communication) Loophole

What if the measurement devices could communicate their settings faster than Aspect's switches could change them? What if the choice of measurement setting was somehow predetermined by the source ("freedom-of-choice")?

2. Detection (or Fair-Sampling) Loophole

Detectors are not 100% efficient. What if the detected pairs are not a fair sample of all pairs? Perhaps the detectors are biased to only click for pairs that agree with quantum predictions.

Experiment 2: Closing All Loopholes (2015)

In 2015, three independent groups (at **TU Delft**, **NIST**, and the **University of Vienna**) performed "loophole-free" Bell tests, closing both loopholes simultaneously.

The Delft Experiment (Hensen et al.)

They entangled the electron spin of two nitrogen-vacancy (NV) centers in diamonds **1.3 km** apart.

- High-speed quantum random number generators ensured "freedom of choice".
- The large separation and precise timing ensured locality.
- Highly efficient detection schemes closed the detection loophole.

Their results violated the Bell inequality with high statistical significance.

The verdict is in: Our universe is non-local. Local Realism is experimentally falsified.

Timeline: Key Entanglement Milestones (1935-1982)

1935 – The EPR Paradox Einstein, Podolsky, and Rosen publish their thought experiment, arguing quantum mechanics is "incomplete" for violating **local realism**.

1964 – Bell's Theorem John Bell devises **Bell's Inequality**, showing that local realism and quantum mechanics give different, testable predictions.

1972 – First Bell Test (Freedman & Clauser) The first experimental test using entangled photons. Results show a clear violation, providing the first evidence against local realism.

1982 – Closing the Locality Loophole (Aspect et al.) Aspect's experiment uses fast switches to close the **"locality loophole"**.

Timeline: Key Entanglement Milestones (1997-Present)

1997 – Quantum Teleportation Anton Zeilinger's group demonstrates quantum teleportation, highlighting entanglement as a key resource for quantum information.

2015 – The Final Word: Loophole-Free Bell Tests Three independent teams finally close all major loopholes simultaneously, providing definitive, unambiguous proof that our universe violates local realism.

2016+ – The Quantum Internet The Chinese "Micius" satellite distributes entangled photons over 1,200 km, demonstrating the feasibility of a global quantum communication network.

The Trillion Dollar Question: Can We Signal Faster Than Light?

The instantaneous collapse suggests FTL influence. Can we exploit this to send signals and violate causality?

NO!

The No-Communication Theorem

It is impossible to use shared quantum entanglement to transmit information faster than the speed of light.

Why No FTL? Alice's Randomness

The first key reason is that Alice cannot choose the outcome of her measurement.

- Consider the Bell state $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.
- When Alice measures, she gets '0' with 50% probability and '1' with 50% probability.
- She cannot force the outcome to be '1' just to send a "1 bit" to Bob.
- Her measured data stream, e.g., "011010...", is fundamentally random and contains no message she encoded.

While her random string is perfectly correlated with Bob's random string, her string by itself is meaningless.

A Broader View: The Density Matrix ρ

The state vector $|\psi\rangle$ is for perfectly known systems (**pure states**). For subsystems or noisy systems, we need the density matrix, ρ.

Pure State: System in a definite state $|\psi\rangle$.

$$\rho = |\psi\rangle\langle\psi|$$

Mixed State: A statistical ensemble of states $\{|\psi_i\rangle\}$ with probabilities p_i.

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

Example: The maximally mixed state (50% $|0\rangle$, 50% $|1\rangle$) is

$$\rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| = \frac{1}{2}I$$

Describing Subsystems: The Partial Trace

What is the state of a single particle that is part of an entangled pair?

- We cannot assign an independent state vector $|\psi\rangle_A$ to it.
- To find the state of Alice's qubit alone (ρ_A), we "trace out" or average over all possibilities for Bob's system. This is the partial trace, Tr_B.

$$\rho_A = \mathrm{Tr}_B(\rho_{AB}) = \sum_k \langle k_B| \rho_{AB} |k_B\rangle$$

This gives us Alice's local state, completely independent of what happens at Bob's end.

Entanglement and Mixedness: A Profound Insight

Let's calculate the local state for Alice's qubit in the pure, entangled Bell state $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

The density matrix for the pair is $\rho_{AB} = |\Phi^+\rangle \langle\Phi^+|$. Applying the partial trace:

$$\begin{aligned}\rho_A &= \langle 0_B|\, \rho_{AB}\, |0_B\rangle + \langle 1_B|\, \rho_{AB}\, |1_B\rangle \\ &= \frac{1}{2}\, |0_A\rangle \langle 0_A| + \frac{1}{2}\, |1_A\rangle \langle 1_A| \\ &= \frac{1}{2} I\end{aligned}$$

Profound Insight

Even though the global state ρ_{AB} is pure (maximal knowledge), Alice's local state ρ_A is **maximally mixed** (minimal knowledge)! For an entangled system, information is in the correlations, not the individual parts.

Why No FTL? Bob's Unchanged Statistics

Let's use the density matrix to prove that Alice's actions don't change Bob's local reality.

- Bob's local state is his reduced density matrix, $\rho_B = \mathrm{Tr}_A(\rho_{AB})$.
- For any Bell state, Bob's initial state is maximally mixed: $\rho_B = \frac{1}{2}I$. He will measure '0' or '1' with 50/50 probability.
- When Alice measures, the global state ρ_{AB} collapses.
- However, if we average over all of Alice's possible (and random) outcomes, Bob's local density matrix **does not change**.

$$\rho_B^{\text{after Alice's measurement}} = \rho_B^{\text{before}} = \frac{1}{2}I$$

- To Bob, who is isolated, the measurement on the other side never happened. His local outcomes remain completely random.

Why No FTL? Correlation vs. Communication

If Bob's results don't change, where is the "spooky action"?

Entanglement creates correlations, not communication.

The non-local connection does not send a message. It ensures the relationship between two random datasets is fixed.

- Alice measures and gets a random string: 01101001
- Bob, far away, measures and gets a random string: 01101001
- Neither can control their string.
- The correlation only becomes apparent when they communicate classically (e.g., by phone) and compare notes. This confirmation is limited by the speed of light.

Experimentally Verifying No FTL Signaling: The Logic

Any long-distance Bell test is also a test of the No-Communication Theorem. The key is to analyze the data from **one location only** before comparing it with the other.

- **Setup:** Alice and Bob are separated by a large distance and share entangled pairs.
- **Alice's Actions:** Alice freely and randomly chooses her measurement settings for each particle she receives.
- **Bob's Local Data:** Bob monitors his stream of measurement outcomes.
- **The Crucial Test:** Can Bob tell from his local statistics alone whether Alice is measuring, what settings she is using, or what her results are?

Experimentally Verifying No FTL Signaling: The Result

The Unanimous Result

In every experiment performed to date, the answer is a resounding **NO**. Bob's data stream is statistically identical to random noise, regardless of what Alice does. The "spooky" correlations only become visible after Alice and Bob use a classical channel (limited by light speed) to compare their separate, random-looking lists of results. This confirms that entanglement upholds causality.

Let's look at three examples.

Example 1: The Delft Test (2015) - Setup

This landmark experiment provided definitive proof against local realism. **Physical System:**

- Two electron spins, each trapped in a **Nitrogen-Vacancy (NV) center** within separate diamonds.
- The labs were nicknamed 'Alice' and 'Bob'.

Distance:

- The labs were separated by **1.3 kilometers** across the university campus.

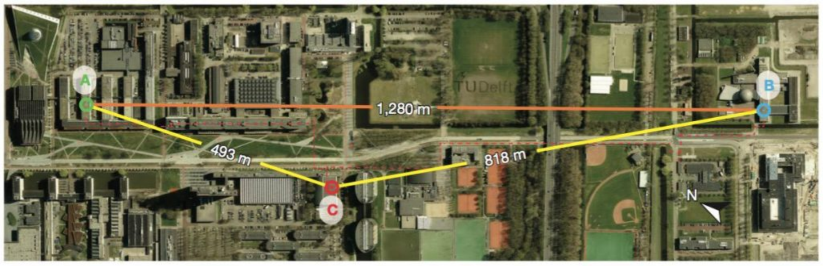


Figure: A map showing the 1.3 km separation.[5]

Example 1: The Delft Test (2015) – No-Communication Proof

The No-Communication Demonstration:

- At Bob's lab, the sequence of spin measurements (up, down, up, up, down...) was recorded.
- When analyzed in isolation, this data stream was statistically random, conforming to a 50/50 probability.
- There was no change or pattern in Bob's data that correlated with when or how Alice chose to measure her diamond's spin. Bob could not tell from his end if Alice was even participating in the experiment.
- The famous Bell inequality violation only appeared after the experiment when the two random-looking data logs were classically compared.

Example 2: The Micius Satellite (2016-Present) – Setup

The Chinese "Micius" quantum satellite took entanglement experiments to an astronomical scale.[6]

Physical System:

- The satellite generates pairs of **polarization-entangled photons**.
- It beams each photon from a pair to two different ground observatories.

Distance:

- The ground stations (e.g., Delingha and Lijiang) were separated by over **1,200 kilometers**.

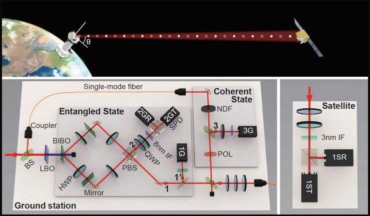


Figure: Micius distributing entangled photons.

Example 2: The Micius Satellite – No-Communication Proof

The No-Communication Demonstration:

- An observatory, say Delingha, receives its stream of photons and measures their polarization.
- The resulting data (e.g., horizontal, vertical, vertical...) shows no inherent message or pattern. It is consistent with random chance.
- The data at Delingha shows no dependence on the measurement choices made at the Lijiang station over 1200 km away.
- The strong quantum correlations were only confirmed after the data from both ground stations were collected and compared via conventional internet.

Example 3: Quantum Teleportation – The Protocol

Quantum teleportation, demonstrated by Anton Zeilinger's group, perfectly illustrates the no-communication principle.

Physical System:

- Alice is given a photon ('Charlie') in an unknown quantum state she wants to "teleport".
- Alice and Bob also share a separate pair of entangled photons.

The Action:

- Alice performs a joint measurement on her two photons (Charlie and her half of the entangled pair).
- This action instantly changes the state of Bob's photon to be related to Charlie's original state.

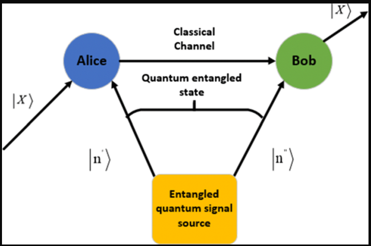


Figure: A schematic of quantum teleportation.

Example 3: Quantum Teleportation – No-Communication Proof

The No-Communication Demonstration:

- Alice's measurement instantly changes Bob's photon.
- However, Bob has no idea this has happened. To him, his photon's state, if measured, would yield a completely random result.
- For Bob to recover the original state of photon Charlie, Alice **must send him two classical bits of information** about the result of her measurement.
- This classical message is limited by the speed of light. Without it, the "teleported" information is inaccessible and useless.

Measuring Uncertainty: Von Neumann Entropy

We can quantify the uncertainty or "mixedness" of a state ρ using **Von Neumann Entropy**.
For a state ρ with eigenvalues λ_i, the entropy is:

$$S(\rho) = -\mathrm{Tr}(\rho \log_2 \rho) = -\sum_i \lambda_i \log_2 \lambda_i$$

- For any **pure state**, $S(\rho_{\text{pure}}) = 0$. (Zero uncertainty).
- For any **mixed state**, $S(\rho) > 0$.
- The maximally mixed state of a qubit, $\rho = \frac{1}{2}I$, has $S(\rho_{\text{mixed}}) = 1$. (Maximum uncertainty).

Quantifying Connection: Entropy of Entanglement

We can now give a precise, quantitative definition of entanglement for a pure two-part state $|\Psi\rangle_{AB}$.

Entropy of Entanglement

The entanglement of $|\Psi\rangle_{AB}$ is the Von Neumann entropy of either of its reduced density matrices:

$$E(|\Psi\rangle) = S(\rho_A) = S(\rho_B)$$

(It can be proven that $S(\rho_A)$ always equals $S(\rho_B)$).

- **Separable state**: For $|00\rangle$, $\rho_A = |0\rangle\langle 0|$ is pure, so $S(\rho_A) = 0$. **Zero entanglement**.
- **Entangled state**: For a Bell state, $\rho_A = \frac{1}{2}I$ is maximally mixed, so $S(\rho_A) = 1$.

A Bell state contains exactly one "e-bit" of entanglement.

Advanced Trickery: Entanglement Swapping

Entanglement can be "teleported" onto particles that have never interacted.

The Protocol:

1. Create two independent entangled pairs (1-2 and 3-4). Alice has 1, Bob has 4.
2. Particles 2 and 3 are brought to a central station (Charlie).
3. Charlie performs a joint **Bell State Measurement (BSM)** on particles 2 and 3.
4. Depending on Charlie's outcome, particles 1 and 4 are instantly projected into an entangled state.
5. Charlie sends his result classically to Alice and Bob, so they know which entangled state they now share.

The Indistinguishability Puzzle

For **distinguishable** particles (e.g., atom A and atom B), entanglement is straightforward. The state

$$|\Psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|\uparrow\rangle_A \otimes |\downarrow\rangle_B + |\downarrow\rangle_A \otimes |\uparrow\rangle_B)$$

has a clear meaning: Alice's spin is correlated with Bob's spin. This relies on being able to **label** the particles (A and B).

The Problem with Identical Particles

Identical particles (e.g., two electrons or two photons) are fundamentally indistinguishable. You cannot label one as "particle 1" and the other as "particle 2". The question "Which electron is in state $|\uparrow\rangle$?" is **meaningless**.

The Conceptual Conflict

The Core Issue

Treating individual, identical particles as the subsystems to be entangled creates a conceptual contradiction. The mathematics used (the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$) is built on distinguishability, which conflicts with the physical principle of indistinguishability.

The modern view is that we don't entangle the identical particles themselves. We entangle the **modes** the particles can occupy.

The Solution: Entangling Modes

- **Modes** are distinguishable, orthogonal single-particle states. They are the containers, not the content.
- Examples of distinguishable modes:
  - The "Horizontal Polarization" mode vs. the "Vertical Polarization" mode.
  - The "Left Path" mode vs. the "Right Path" mode.

  These modes are our new "Alice" and "Bob".
- **Fock Space** is the framework. We track the occupation number of each mode ($|n_A, n_B, \ldots\rangle$). This inherently handles indistinguishability.

Mode Entanglement: Example

Example: Two-Photon Mode Entanglement

Consider two distinguishable spatial modes, Path A and Path B. An entangled state is:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|2_A, 0_B\rangle + |0_A, 2_B\rangle)$$

Interpretation: The system is in a superposition where either both photons took Path A, or both took Path B. The entanglement is a correlation between the **occupation numbers of the paths**.

Mode Entanglement: Analogy

Classical Analogy: Two Water Pipes

- Imagine two separate pipes, A and B (the modes). The water molecules (the particles) are identical.
- A "mode-entangled" analogue: We know the total energy of waves in the system is fixed.
- If we measure a high-energy wave in Pipe A, we instantly know there's a low-energy wave in Pipe B.
- The correlation is between the properties of the **pipes**, not individual, untraceable water molecules.

The Jaynes-Cummings Model (JCM)

The JCM is the simplest, yet fully quantum, model of light-matter interaction. It describes a single **two-level system** (qubit) interacting with a single mode of a **quantized harmonic oscillator** (photon/phonon).[7]
It is the "hydrogen atom" for quantum optics and the foundation of Cavity QED, Circuit QED, and Trapped Ion physics.

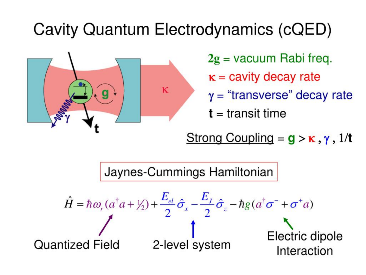


Figure: A two-level atom inside an optical cavity.

The Players in the Model

1. A Two-Level Atom

- Two energy levels: Ground $|g\rangle$ and Excited $|e\rangle$.
- Transition frequency: ω_a.
- Pauli operators:

$$\hat{\sigma}_+ = |e\rangle \langle g|$$

$$\hat{\sigma}_- = |g\rangle \langle e|$$

2. A Single-Mode Field

- A cavity supporting light of frequency ω_c.
- Energy quanta are **photons**.
- State is photon number, $|n\rangle$.
- Boson operators:

$$\hat{a}^\dagger |n\rangle \propto |n+1\rangle$$

$$\hat{a} |n\rangle \propto |n-1\rangle$$

The Jaynes-Cummings Hamiltonian

The total energy is $\hat{H}_{\text{JC}} = \hat{H}_{\text{atom}} + \hat{H}_{\text{field}} + \hat{H}_{\text{int}}$.

1. Free Evolution

$$\hat{H}_{\text{atom}} = \frac{1}{2}\hbar\omega_a\hat{\sigma}_z \quad , \quad \hat{H}_{\text{field}} = \hbar\omega_c\hat{a}^\dagger\hat{a}$$

2. Interaction

The term describing the exchange of energy.

$$\hat{H}_{\text{int}} = \hbar g(\hat{a}^\dagger\hat{\sigma}_- + \hat{a}\hat{\sigma}_+)$$

Here, g is the coupling strength.

- $\hat{a}^\dagger\hat{\sigma}_-$: Atom de-excites, creating a photon (**emission**).
- $\hat{a}\hat{\sigma}_+$: Atom excites, absorbing a photon (**absorption**).

This term conserves the total number of excitations.

Dynamics (1): Vacuum Rabi Oscillations

Consider an excited atom in an empty cavity.

- Initial State ($t = 0$): $|\psi(0)\rangle = |e, 0\rangle$ (a separable state).
- The interaction term couples this state to $|g, 1\rangle$.

The state at time t (on resonance) is:

$$|\psi(t)\rangle = \cos(gt)\,|e, 0\rangle - i\sin(gt)\,|g, 1\rangle$$

The probability of finding the atom excited oscillates as $\cos^2(gt)$. This coherent, reversible exchange of a single energy quantum is a fundamental signature of quantum light-matter interaction.

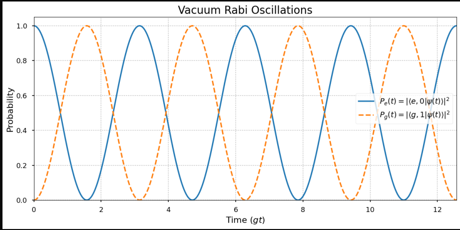


Figure: Vacuum Rabi Oscillations

Generating Entanglement with the JCM

The state during Rabi oscillations is:

$$|\psi(t)\rangle = \cos(gt)\,|e,0\rangle - i\sin(gt)\,|g,1\rangle$$

- At $t = 0$, the state is $|e,0\rangle$ (separable).
- For any time t where both $\cos(gt)$ and $\sin(gt)$ are non-zero, this state is entangled.
- The interaction Hamiltonian, $\hat{H}_{\text{int}}$, is the **entangling operation**.

Creating a Bell State

If we stop the evolution at a time t_0 such that $gt_0 = \pi/4$:

$$\begin{aligned} |\psi(t_0)\rangle &= \cos(\pi/4)\,|e,0\rangle - i\sin(\pi/4)\,|g,1\rangle \\ &= \frac{1}{\sqrt{2}}(|e,0\rangle - i\,|g,1\rangle) \end{aligned}$$

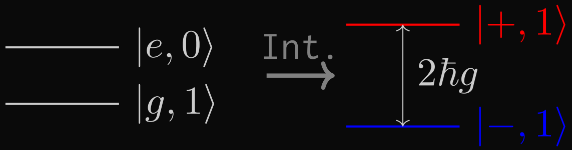
This is a **maximally entangled state** between the atom and the cavity field. The JCM interaction naturally generates entanglement.

The Dressed States Picture

The "bare states" $|e,0\rangle$ and $|g,1\rangle$ are not the true energy eigenstates of the interacting system. The true eigenstates are the "dressed states".

The dressed states are maximally entangled superpositions:

$$|+,1\rangle = \frac{1}{\sqrt{2}}(|e,0\rangle + |g,1\rangle)$$
$$|-,1\rangle = \frac{1}{\sqrt{2}}(|e,0\rangle - |g,1\rangle)$$



The interaction lifts the degeneracy of the bare states. This energy splitting is the **Vacuum Rabi Splitting**.

Dynamics (2): Collapse and Revival

If the atom interacts with a coherent state $|\alpha\rangle = \sum_n c_n |n\rangle$, the dynamics is a sum of Rabi oscillations for each photon number n, each with frequency $\Omega_n = 2g\sqrt{n+1}$.

- **Collapse:** Since the frequencies Ω_n are different, they quickly dephase, and the total oscillation amplitude "collapses".
- **Revival:** Remarkably, after a certain time, the discrete nature of the frequencies allows them to rephase, and the oscillation revives!

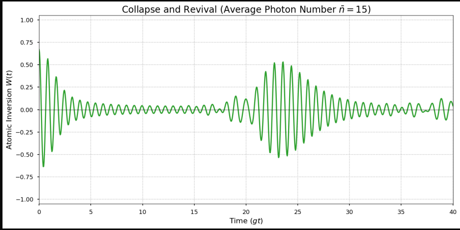


Figure: Visualization of Revival and Collapse of a Wavefunction

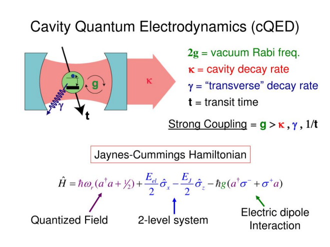
The revival is direct proof of the quantized nature of the field.

The Jaynes-Cummings Model in the Lab

The JCM is experimentally realized in many cutting-edge systems.

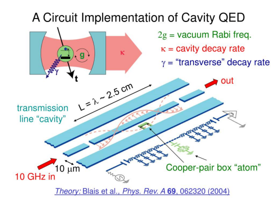
Cavity QED:

- Single atoms coupled to optical photons in a cavity made of super-mirrors.



Circuit QED:

- Superconducting artificial atoms ("transmons") coupled microwave photons in a resonator on a chip.



The JCM Beyond Cavities: Trapped Ions

The JCM describes any two-level system coupled to a harmonic oscillator.

Trapping an Ion:

- A single ionized atom (e.g., Ca^+) is confined by electric fields in a **Paul Trap**.

The Harmonic Oscillator:

- The ion oscillates in the trap.
- This motion is quantized, and the energy quanta are called phonons.

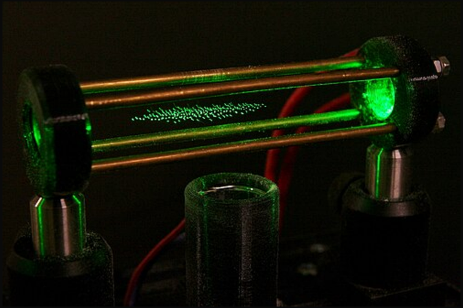


Figure: Charged flour grains held in a Paul ion trap. The grains are glowing because they are being illuminated with green light

JCM in a Trapped Ion: The Analogy

We can make a direct mapping from Cavity QED to a trapped ion.

The Analogy

Cavity QED		**Trapped Ion**
Atom's electronic states	$\Longleftrightarrow$	Ion's internal electronic states
Cavity **photons**	$\Longleftrightarrow$	Quantized motion (**phonons**)
Atom-photon coupling	$\Longleftrightarrow$	Coupling via tuned lasers

The **Qubit** is two stable internal electronic levels of the ion ($|\downarrow\rangle, |\uparrow\rangle$). The **Bosonic Mode** is a vibrational mode of the ion in the trap.

Overall Conclusion

- **Entanglement** reveals the non-local nature of quantum reality. It is a quantifiable resource that fuels quantum technologies.
- Decades of experiments have definitively shown that our universe violates **local realism**, confirming the predictions of quantum mechanics.
- The **Jaynes-Cummings Model** is the fundamental theory for engineering the interaction between qubits and bosons (light or motion), teaching us how to generate and control entanglement.

The Big Picture

By applying the physics of the JCM in platforms like circuit QED and trapped ions, we can generate, manipulate, and utilize the resource of entanglement. Together, these concepts form the foundation of the second quantum revolution.

Appendix: Schmidt Decomposition

Any bipartite pure state $|\psi\rangle_{AB}$ can be written as:

$$|\psi\rangle_{AB} = \sum_{i=1}^{k} \lambda_i |u_i\rangle_A |v_i\rangle_B$$

where λ_i are the **Schmidt coefficients** and $\{|u_i\rangle\}$, $\{|v_i\rangle\}$ are orthonormal bases. The number of non-zero terms k is the Schmidt rank.

Connection to Entanglement:

- A state is **separable** iff its Schmidt rank is 1.
- A state is **entangled** iff its Schmidt rank is > 1.
- The entanglement entropy is computed from the coefficients:

$$E(|\psi\rangle) = -\sum_i \lambda_i^2 \log_2(\lambda_i^2)$$

Example: For $|\Phi^+\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$, the coefficients are $\lambda_1 = \lambda_2 = 1/\sqrt{2}$. The rank is 2 (entangled) and the entropy is 1.

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Thank You!

Questions?